An Overlapping Generations Model with Capital Accumulation and its Implications for Interest Rate Theory

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Abstract

The Diamond (1965) and Tirole (1985) overlapping generations models assume capital is consumed each period after being used in production rather than accumulate for future productive use. Neither Diamond nor Tirole explicitly model the consumption of capital. After explicitly modeling the consumption of capital and examining the implications, we present and examine an alterantive model where non-depreciated capital accumulates. We find that allowing capital to accumulate fundamentally alters the model. Specifically, we find capital market clearing does not generally imply product market clearing when capital accumulates, unless the interest rate paid on saving differs from the capital rental rate. Thus, modifying the model to include capital accumulation implies we must modify the interest rate theory obtained from the model.

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1 Introduction

In introducing his pure consumption version of the overlapping generations (OG) model, Samuelson (1958, p. 467) expressed his purpose was to "give a complete general equilibrium solution to the determination of the time-shape of interest rates." Fisher (1930) had earlier provided a theory of interest that concluded "impatience" would make interest rates positive. Gale (1973) questioned Fisher's conclusion and offered an extension of Samuelson (1958) model to elucidate his concern. Gale emphasized that, regardless of the degree of impatience or investment opportunities, the age demographics of the population, young versus old, will also influence whether there is a surplus or shortage of saving. Samuelson (1958)'s "biological theory of interest" directly recognizes the impact of age demographics. Gale not only helped us appreciate that no single factor will tend to determine the interest rate level, but also he taught us we should not expect to understand interest rate determination fully if we do not consider overlapping generations.

Diamond (1965) extended the Samuelson (1958) pure consumption OG economy to include production, and Tirole (1985) extended Diamond's production economy to allow bubbles. The Diamond extension provides the opportunity to understand how the use of saving to finance capital influences the interest rate level, and the Tirole extension provides the opportunity to understand how the development of bubbles influences the interest rate level. Yet, both Diamond and Tirole assume "the equilibrium interest rate will equal the marginal product of capital" (Diamond, 1965, p. 1130), which also equals the captial rental rate because competitive factor markets are assumed. Thus, in the Diamond-Tirole framework, understanding interest rate determination amounts to identifying the factors that determine the productivity of capital. Samuelson's biology (i.e, the population growth rate), Fisher's impatience, bubbles, and other factors may play a role in determining the interest rate, but they do so by determining the level of capital employed, which determines the productivity of capital, which determines the capital rental rate, which is assumed to equal to the interest rate.

Our work here is related to the work of Aiyagari (1992), who shows Walras' Law need not hold in a pure consumption economy. The Diamond (1965) and Tirole (1985) economies are not pure consumption economies, but will Walras' Law necessarily hold? Because product can be saved and then invested, there are two markets to consider in the Diamond and Tirole economies: The capital market and product market. Diamond and Tirole each explicitly impose capital market clearing, but neither explicitly impose product market clearing, nor do either explicitly show that capital market clearing implies product market clearing. We begin by examining the the original Diamond (1965) and Tirole (1985) economies, and we show the capital market clearing may or may not imply product market clearing. In particular, we the Diamond and Tirole models are general equilibrium models where all markets clear, as long as any non-depreciated capital is consumed by the old as part of the return promised on saving. However, we also show capital market clearing does not imply product market clearing if the non-depreciated capital is consumed by either young or old under a contract separate from the promised return on saving.

We then examine an economy where capital can depreciate, but non-depreciated capital remains with the firm and accumulates, rather than being consumed. Interestingly, we find that allowing capital to accumulate significantly impacts the model. In particular, when we restrict the interest rate to be equal to the rental rate on capital as in the Diamond and Tirole economies, we find that capital market clearing implies product market clearing in only two special cases. One special case is where capital entirely depreciates; the other is where the economy begins and remains in the golden rule steady state. We show that allowing the interest rate paid on saving to deviate from the capital rental rate allows the product market to clear when the capital market clears.

Because the interest rate must deviate from the capital rental rate for a general equilibrium to occur in all cases, we find that modifying the model to assume capital accumulates requires us to modify our theory of interest rate determination. The interest rate is still related to capital and capital rental rate, but there will generally be a gap between the two. The rate of depreciation and the degree to which a bubble forms each influence this gap.

2 The Tirole-Diamond Economy

Consider an OG economy equivalent to that of Tirole (1985), which includes the Diamond (1965) economy as a special case.

The generation t consumer maximizes utility $U = U(c_t^y, c_{t+1}^o)$ by optimally choosing young and old age consumption levels c_t^y and c_{t+1}^o, along with the saving level s_t, subject to the young age and old age budget constraints

$$c_t^y = w_t - s_t \tag{1}$$

and

$$c_{t+1}^o = [1 + r_{t+1}]s_t, \tag{2}$$

where w_t is the real wage and r_{t+1} is the real rate of interest earned on saving. Optimization yields the first order condition

$$U_{c^y}(c_t^y, c_{t+1}^o) = [1 + r_{t+1}]U_{c^o}(c_t^y, c_{t+1}^o). \tag{3}$$

Together, conditions $(1) - (3)$ determine the optimal levels of c_t^y, c_{t+1}^o and s_t as functions of the real wage level w_t and the real interest rate level r_{t+1}.

The period t population consists of L_t young age and L_{t-1} old age consumers, and the generation t population grows at rate n, so $L_t = [1 + n]L_{t-1}$. The period t production level Y_t depends upon capital level K_t and labor level L_t, according to $Y_t = F(K_t, L_t)$. Defining capital and output per labor unit as $k_t = \frac{K_t}{L_t}$ and $y_t = \frac{Y_t}{L_t}$, the assumption that production exhibits constant returns to scale, along with the assumption of diminishing returns to each input, implies output per labor unit can be presented as

$$y_t = f(k_t), \tag{4}$$

where $f'(k_t) > 0$ and $f''(k_t) < 0$. Firm profit maximization with respect to capital and labor implies the rental rate paid on capital is

$$r_t = f'(k_t) - \delta, \tag{5}$$

and the wage rate paid on labor is

$$w_t = y_t - [r_t + \delta]k_t. \tag{6}$$

The optimization conditions (5) and (6) are of special interest. Diamond (1965, p. 1127) explicitly assumes "there is no depreciation, and that, since capital and output are the same commodity, one can consume one's capital." Tirole (1985) follows Diamond and also adopts this assumption. Tirole also adopts the (Diamond, 1965, p. 1130) assumption that "the equilibrium interest rate will equal the marginal product of capital," which also equals the capital rental rate when there is no depreciation. This explains why the variable r is used in both condition (5) and condition (2). We use the "net of depreciation" optimization conditions because we desire to consider the general case where depreciation may or may not occur.[1]

In the Diamond economy, the saving of young generation t consumers may only flow into a capital backed asset, but in the more general Tirole economy it may also flow into a

[1] See Sala-i Martin and Barro (1995, p. 69) for a good description of the firm optimization problem with depreciation.

bubbly asset. For simplicity, Tirole assumes the bubble asset pays the same rate of interest r_{t+1} as the capital backed asset, implying the two assets are equally risky, neither being risky. For the bubble asset to be held, the real value of the bubble must grow at a rate just sufficient to provide the promised rate of return, which implies $B_{t+1} = [1 + r_{t+1}]B_t$. The bubble per labor unit $b_t = B_t/L_t$ must therefore grow according to

$$b_{t+1} = \frac{[1 + r_{t+1}]}{[1 + n]} b_t. \tag{7}$$

The saving not flowing into the bubbly asset flows into the capital backed asset as investment. Investment accumulates as capital according to $I_t = K_{t+1} - [1 - \delta]K_t$, where δ is the capital depreciation rate. The capital market clears when the supply of saving S_t equals the demand $K_{t+1} - [1 - \delta]K_t + B_t$, or on a per labor unit basis when

$$s_t = [1 + n]k_{t+1} + b_t. \tag{8}$$

Together, the eight equations $(1) - (8)$ determine the paths of the eight endogenous variables $s_t, c_t^y, c_{t+1}^o, y_t, r_t$, w_t, k_{t+1}, and b_{t+1}. The variables k_t and b_t are predetermined, and the variables n and δ are exogenous.

Neither Diamond (1965) model nor Tirole (1985) explicitly impose product market clearing, though Diamond does present the product market clearing condition at one point. The product market clears when the supply of output $Y_t + [1 - \delta]K_t$ equals the demand $L_t c_t^y + L_{t-1} c_t^o + K_{t+1}$ for period t consumption and period $t + 1$ capital. Thus, on a per labor unit basis, the product market clears when

$$y_t + [1 - \delta]k_t = c_t^y + \frac{c_t^o}{[1 + n]} + [1 + n]k_{t+1}. \tag{9}$$

A key question for us is whether capital market clearing implies product market clearing. The economy only contains these two markets, so Walras' Law would indicate that product market should clear if the capital market clears. However, in a pure consumption context, Aiyagari (1992) has shown that Walras's Law will not necessarily hold for an OG model. We now explore this issue for the Diamond-Tirole economy we have presented.

3 Product Market Clearing in the Diamond-Tirole Model

When Diamond indicates capital left over from production is consumed, he does not specify the contractual arrangement. There seem to be three cases. First, and least likely, the capital could be consumed by the young, supplementing the wage to finance young age consumption. Second, and more likely, the capital could be consumed by the old, financing old age consumption separate from the return promised on saving. Finally, and perhaps most likely, the capital could be consumed as part of the return promised on saving. For each case, we now examine the following question: Does capital market clearing imply product market clearing, as would be implied by Walras' Law?

If the leftover capital $[1-\delta]K_t$ is consumed by the young, then each young consumer receives an additional $[1-\delta]k_t$ and the young age budget constraint (1) becomes $c_t^y = w_t - s_t + [1-\delta]k_t$. The remaining conditions of the Diamond-Tirole model remain unchanged. Does capital market clearing imply product market clearing for this first case?

Beginning with the capital market clearing condition (8), we use our now modified young age budget constraint to eliminate saving variable s_t and obtain $w_t + [1-\delta]k_t - c_t^y = [1+n]k_{t+1} + b_t$. Then, using (6) to eliminate w_t, we have $y_t - [r_t+\delta]k_t + [1-\delta]k_t - c_t^y = [1+n]k_{t+1} + b_t$. Adding $c_t^y + \frac{c_t^o}{1+n}$ to both sides and rearranging, we obtain

$$\frac{c_t^o}{1+n} - [r_t+\delta]k_t - b_t = c_t^y + \frac{c_t^o}{1+n} + [1+n]k_{t+1} - [1-\delta]k_t - y_t. \tag{10}$$

Examining the product market clearing condition (9), we observe that the right side of condition (10) is the economy's excess demand for output. Thus, we find capital market clearing implies product market clearing if and only if $\frac{c_t^o}{1+n} - [r_t+\delta]k_t - b_t = 0$.

To examine whether this last condition always holds, we first use condition (2) to eliminate the old age consumption c_t^o and obtain $\frac{c_t^o}{1+n} - [r_t+\delta]k_t - b_t = \left[\frac{1+r_t}{1+n}\right] s_{t-1} - [r_t + \delta]k_t - b_t$. Using (8) to eliminate the savings variable, s_{t-1}, we obtain $\frac{c_t^o}{1+n} - [r_t+\delta]k_t - b_t = \left[\frac{1+r_t}{1+n}\right][[1+n]k_t + b_{t-1}] - [r_t+\delta]k_t - b_t$. Then, using condition (7) to eliminate $\left[\frac{1+r_t}{1+n}\right] b_{t-1}$, we obtain $\frac{c_t^o}{1+n} - [r_t+\delta]k_t - b_t = [1+r_t]k_t + b_t - [r_t+\delta]k_t - b_t$. That is, we obtain $\frac{c_t^o}{1+n} - [r_t+\delta]k_t - b_t = [1-\delta]k_t$. Since $[1-\delta]k_t = 0$ only for the special case where capital entirely depreciates(i.e., $\delta = 1$), we find capital market clearing does *not* imply product market clearing, in general, when the non-depreciated capital is consumed by the young.

If the leftover capital $[1-\delta]K_{t+1}$ is consumed by the old in period $t+1$, separate from the promised return on saving, then each old consumer receives an additional $[1-\delta][1+$

$n]k_{t+1}$ and the old age budget constraint (2) becomes $c^o_{t+1} = [1+r_{t+1}]s_t+[1-\delta][1+n]k_{t+1}$. The remaining conditions of the Diamond-Tirole model remain unchanged.

To examine whether capital market clearing implies product market clearing for this second case, we again begin with the capital market clearing condition (8) and use the young age budget constraint (1) to eliminate the saving variable s_t, here obtaining $w_t - c^y_t = [1+n]k_{t+1} + b_t$. Then, using (6) to eliminate w_t, we have $y_t - [r_t+\delta]k_t - c^y_t = [1+n]k_{t+1} + b_t$. Adding $c^y_t + \frac{c^o_t}{1+n} - [1-\delta]k_t$ to both sides and rearranging, we obtain $\frac{c^o_t}{1+n} + y_t - [r_t+\delta]k_t - [1-\delta]k_t = c^y_t + \frac{c^o_t}{1+n} + [1+n]k_{t+1} - [1-\delta]k_t + b_t$, or

$$\frac{c^o_t}{1+n} - [1+r_t]k_t - b_t = c^y_t + \frac{c^o_t}{1+n} + [1+n]k_{t+1} - [1-\delta]k_t - y_t. \quad (11)$$

The right side of condition (11) is the economy's excess demand for output. Thus, we find that the product market clears if and only if $\frac{c^o_t}{1+n} - [1+r_t]k_t - b_t = 0$.

To examine whether this last equation always holds, we first use our revised condition (2) to eliminate the old age consumption c^o_t, obtaining $\frac{c^o_t}{1+n} - [1+r_t]k_t - b_t = \left[\frac{[1+r_t]s_{t-1}+[1-\delta][1+n]k_t}{1+n}\right] - [1+r_t]k_t - b_t$ Using (8) to eliminate the savings variable, s_{t-1}, we obtain $\frac{c^o_t}{1+n} - [1+r_t]k_t - b_t = \left[\frac{[1+r_t][[1+n]k_t+b_{t-1}]+[1-\delta][1+n]k_t}{1+n}\right] - [1+r_t]k_t - b_t$. Using condition (7) to eliminate $\left[\frac{1+r_t}{1+n}\right] b_{t-1}$, we obtain $\frac{c^o_t}{1+n} - [1+r_t]k_t - b_t = [1+r_t]k_t + [1-\delta]k_t + b_t - [1+r_t]k_t - b_t$. That is, we obtain $\frac{c^o_t}{1+n} - [1+r_t]k_t - b_t = [1-\delta]k_t$. Thus, we find capital market clearing generally does *not* imply product market clearing when the non-depreciated capital is consumed by the old separate from the return promised on saving. The implication again only occurs in the complete depreciation special case where $\delta = 1$.

If the leftover capital $[1-\delta]K_{t+1}$ is consumed by the old in period $t+1$ as part of the promised return on saving, the conditions of the Diamond-Tirole model remain unchanged. As we seek to examine whether capital market clearing implies product market clearing for this third case, the steps are the same as in the second case up to reaching condition (11). Thus, we again find that the product market clears if and only if $\frac{c^o_t}{1+n} - [1+r_t]k_t - b_t = 0$.

To examine whether this last equation always holds, we first use condition (2) to eliminate the old age consumption c^o_t, obtaining $\frac{c^o_t}{1+n} - [1+r_t]k_t - b_t = \left[\frac{1+r_t}{1+n}\right] s_{t-1} - [1+r_t]k_t - b_t$ Using (8) to then eliminate the savings variable, s_{t-1}, we obtain $\frac{c^o_t}{1+n} - [1+r_t]k_t - b_t = \left[\frac{1+r_t}{1+n}\right] [[1+n]k_t + b_{t-1}] - [1+r_t]k_t - b_t$. Using condition (7) to eliminate $\left[\frac{1+r_t}{1+n}\right] b_{t-1}$, we obtain $\frac{c^o_t}{1+n} - [1+r_t]k_t - b_t = [1+r_t]k_t + b_t - [1+r_t]k_t - b_t$. That is, we obtain $\frac{c^o_t}{1+n} - [1+r_t]k_t - b_t = 0$. Thus, we find capital market clearing *does* imply product market clearing when the non-depreciated capital is consumed by the old separate from the return promised on saving.

4 Modifying the Model to Assume Capital Accumulates

When we modify the Diamond-Tirole model to allow for capital accumulation, all of the model conditions remain the same except the capital market clearing condition. When capital is consumed, saving in period t must finance all of period $t+1$ capital (in addition to financing the period t bubble), so $S_t = K_{t+1} + B_t$, as noted above. However, when capital can accumulate, period t saving need finance only the addition to period $t+1$ capital (in addition to financing the period t bubble), so $S_t = K_{t+1} - [1-\delta]K_t + B_t$. That is, on a per young consumer basis, the capital market clearing condition (8) becomes

$$s_t = [1+n]k_{t+1} - [1-\delta]k_t + b_t. \tag{12}$$

We now explore whether capital market clearing implies product market clearing. Beginning with the capital market clearing condition (12) and use the young age budget constraint (1) to eliminate the saving variable s_t, here obtaining $w_t - c_t^y = [1+n]k_{t+1} - [1-\delta]k_t + b_t$. Then, using (6) to eliminate w_t, we have $y_t - [r_t+\delta]k_t - c_t^y = [1+n]k_{t+1} - [1-\delta]k_t + b_t$. Adding $c_t^y + \frac{c_t^o}{1+n}$ to both sides and rearranging, we obtain

$$\frac{c_t^o}{1+n} - [r_t+\delta]k_t - b_t = c_t^y + \frac{c_t^o}{1+n} + [1+n]k_{t+1} - [1-\delta]k_t - y_t. \tag{13}$$

The right side of condition (13) is the economy's excess demand for output. Thus, we find that the product market clears if and only if $\frac{c_t^o}{1+n} - [r_t+\delta]k_t - b_t = 0$.

To examine whether this last equation always holds, we first use condition (2) to eliminate the old age consumption c_t^o, obtaining $\frac{c_t^o}{1+n} - [r_t+\delta]k_t - b_t = \left[\frac{[1+r_t]s_{t-1}}{1+n}\right] - [r_t+\delta]k_t - b_t$ Using our new capital market clearing condition (12) to eliminate the savings variable, s_{t-1}, we obtain $\frac{c_t^o}{1+n} - [r_t+\delta]k_t - b_t = \left[\frac{[1+r_t][[1+n]k_t - [1-\delta]k_{t-1} + b_{t-1}]}{1+n}\right] - [r_t+\delta]k_t - b_t$. Using condition (7) to eliminate $\left[\frac{1+r_t}{1+n}\right] b_{t-1}$, we obtain $\frac{c_t^o}{1+n} - [r_t+\delta]k_t - b_t = [1+r_t]k_t - \left[\frac{[1+r_t][1-\delta]}{1+n}\right] k_{t-1} + b_t - [r_t+\delta]k_t - b_t$. That is, we obtain $\frac{c_t^o}{1+n} - [r_t+\delta]k_t - b_t = [1-\delta]k_t - \left[\frac{[1+r_t][1-\delta]}{1+n}\right] k_{t-1}$. Thus, we find capital market clearing generally does *not* imply product market clearing. We do find there are two special cases were Walras's Law holds. First, when capital entirely depreciates (i.e., $\delta = 1$). The other special case occurs when the economy is in the Golden Rule steady state, where $r_t = n$ and $k_{t-1} = k_t$ for all t.

However, in all other cases, capital market clearing does not imply product market clearing.

5 A General Equilibrium Overlapping Generations Model with Capital Accumulation

We have just shown that Walras' Law does not hold if the only adjustment to the Diamond-Tirole model is to allow capital to accumulate. We now show we can obtain a general equilibrium for our model with capital accumulation if we relax the assumption that the interest rate paid on saving must be equal to the capital rental rate. Because Walras' Law does not hold for the model with capital accumulation, the product market clearing condition is not redundant. Rather, it can be used, and indeed it seems it should be used, to determine the rate of interest for the economy separate and distinct form the capital rental rate.

If the interest rate paid on savings can deviate from the capital rental rate, then we must adjust a number of the model conditions. Specifically, letting i_t denote the period t interest rate, the old age budget constraint (2) becomes

$$c_{t+1}^o = [1 + i_{t+1}]s_t, \tag{14}$$

The first order condition (3) becomes

$$U_{c^y}(c_t^y, c_{t+1}^o) = [1 + i_{t+1}]U_{c^o}(c_t^y, c_{t+1}^o). \tag{15}$$

Finally, the bubble evolution condition (7) becomes

$$b_{t+1} = \frac{[1 + i_{t+1}]}{[1 + n]} b_t. \tag{16}$$

Together, conditions (1), (4), (5), (6), (9), (12), (14), (15), and (16) determine the paths of the nine endogenous variables $s_t, c_t^y, c_{t+1}^o, y_t, r_t$, i_{t+1}, w_t, k_{t+1}, and b_{t+1}. The variables k_t and b_t are predetermined, and the variables n and δ are exogenous.

We now reduce the model to obtain two basic dynamic equations that determine the equilibrium path for the economy, given the initial conditions k_0 and b_0. Using the young age budget constraint (1) to eliminate saving variable, the capital market clearing condition (12) becomes $w_t - c_t^y = [1 + n]k_{t+1} - [1 - \delta]k_t + b_t$. Using the employment optimization condition (6) to eliminate the wage variable, we obtain $y_t - [r_t + \delta]k_t - b_t - c_t^y = [1 +$

$n]k_{t+1} - [1-\delta]k_t$. Adding $c_t^y + \frac{c_t^o}{[1+n]}$ to both sides and rearranging, we obtain

$$\frac{c_t^o}{[1+n]} - [r_t+\delta]k_t - b_t = c_t^y + \frac{c_t^o}{[1+n]} + [1+n]k_{t+1} - [1-\delta]k_t - y_t \qquad (17)$$

Examining the product market clearing condition (17), we see that the right side is the economy's excess demand for output. The left side is the degree to which the net capital payment of the firm plus the bubble falls short of the promise to the old generation. If it were true that $\frac{c_t^o}{[1+n]} = [r_t+\delta]k_t + b_t$ always holds, then capital market clearing would always imply product market clearing. However, we showed above, it does not always hold, so we now continue to find two basic dynamic equations.

Using old age budget constraint (14) to eliminate c_t^o from our last condition, we note that the excess demand for output can be presented as $\frac{c_t^o}{[1+n]} - [r_t+\delta]k_t - b_t = \frac{[1+i_t]s_{t-1}}{[1+n]} - [r_t + \delta]k_t - b_t$. Using the capital market equilibrium condition (12) for period $t-1$ to eliminate the saving variable, we obtain $\frac{c_t^o}{[1+n]} - [r_t+\delta]k_t - b_t = \frac{[1+i_t][[1+n]k_t - [1-\delta]k_{t-1} + b_{t-1}]}{[1+n]} - [[r_t+\delta]k_t + b_t]$. Using the bubble condition (16) for period $t-1$, we eliminate $\frac{[1+i_t]}{[1+n]}b_{t-1}$ to obtain $\frac{c_t^o}{[1+n]} - [r_t+\delta]k_t - b_t = \frac{[1+r_t][[1+n]k_t - [1-\delta]k_{t-1}]}{[1+n]} + b_t - [[r_t+\delta]k_t + b_t]$, implying the excess demand for output is equal to

$$\frac{c_t^o}{[1+n]} - [r_t+\delta]k_t - b_t = [1+i_t]\left[k_t - \left[\frac{1-\delta}{1+n}\right]k_{t-1}\right] - [r_t+\delta]k_t. \qquad (18)$$

Setting this excess demand for output equal to zero, using condition (5) to replace $r_t + \delta$, incrementing time one period forward, and rearranging we obtain

$$1 + i_{t+1} = \frac{f'(k_{t+1})k_{t+1}}{k_{t+1} - \left[\frac{1-\delta}{1+n}\right]k_t}, \qquad (19)$$

Condition (19) determines the interest rate i_{t+1} as a function of k_{t+1}, k_t, and the parameters δ and n.

Let $s_t(w_t, i_{t+1})$ denote the optimal savings level obtained from conditions (1), (14), and (15). Using this optimum, we can write the capital market clearing condition (12) as $s_t(w_t, i_{t+1}) = [1+n]k_{t+1} - [1-\delta]k_t + b_t$. Using conditions (4),(5), (6) and (19) to eliminate the variables y_t, r_t, w_t and i_{t+1}, we can rewrite the capital market clearing condition (12) and bubble growth condition (16) as

$$s_t\left(f(k_t)-f'(k_t)k_t,\frac{f'(k_{t+1})k_{t+1}}{k_{t+1}-\left[\frac{1-\delta}{1+n}\right]k_t}-1\right)=[1+n][k_{t+1}-[1-\delta]k_t+b_t \quad (20)$$

and

$$b_{t+1}=\left[\frac{f'(k_{t+1})k_{t+1}}{[1+n]k_{t+1}-[1-\delta]k_t}\right]b_t. \quad (21)$$

Conditions (20) and (21) are the basic dynamic equations for our model with capital accumulation. They determine the levels of the core state variables k_{t+1} and b_{t+1} from the previous period predetermined levels k_t and b_t and the exogenous variables n and δ. Given initial capital and bubble levels k_0 and b_0, conditions (20) and (21) recursively determine the paths for k_t and b_t. Given the paths of these core variables, the other equations of the model determine the paths of the other endogenous variables, including the interest rate and capital rental rate.

To obtain a particular example, consider the utility and production functions used by Diamond (1965): $U(c_t^y,c_{t+1}^o)=Bln(c_t^y)+[1-B]ln(c_{t+1}^o)$ and $f(k_t)=Ak_t^\alpha$. Conditions (20) and (21) become

$$k_{t+1}=\frac{[1-B][1-\alpha]}{1+n}Ak_t^\alpha+\frac{1-\delta}{1+n}k_t-\frac{1}{1+n}b_t \quad (22)$$

$$b_{t+1}=\left[\frac{\alpha Ak_{t+1}^\alpha}{[1+n]k_{t+1}-[1-\delta]k_t}\right]b_t. \quad (23)$$

We note that, if $\delta=1$, then the system is identical to the system examined by Tirole (1985). Interestingly, this is also one of the special cases examined above where capital market clearing does imply product market clearing. Thus, we find that the dynamics of our model with capital accumulation are identical to the dynamics of the Diamond-Tirole economy without capital accumulation when capital entirely depreciates. When $0\le\delta<1$, then the dynamics are similar to the dynamics of Tirole's system, though not identical. In particular, as for Tirole's system, the path this system follows depends upon the size of the initial bubble, and the system either converges to the golden rule steady state with a bubble, converges to the Diamond steady state with no bubble, or diverges and breaks down.

In the Appendix, we show that the golden rule steady state generates the capital stock level $k=\left[\frac{\alpha A}{n+\delta}\right]^{\frac{1}{1-\alpha}}$, the bubble $b=[1-B][1-\alpha]A\left[\frac{\alpha A}{n+\delta}\right]^{\frac{\alpha}{1-\alpha}}-[n+\delta]\left[\frac{\alpha A}{n+\delta}\right]^{\frac{1}{1-\alpha}}$, the interest rate $i=n$, and the capital rental rate $r=n$. That is, we find that the interest rate is equal to the capital rental rate in the Golden Rule steady state, as Tirole (1985) found for his economy. In this Golden Rule steady state, the growth rate of labor entirely determines the interest rate and capital rental rate; no other variable matters.

If the economy alternatively converges to the Diamond steady state, where there is no bubble, then $k = \left[\frac{[1-B][1-\alpha]A}{n+\delta}\right]^{\frac{1}{1-\alpha}}$, $b = 0$, $i = \frac{\alpha[1+n]}{[1-B][1-\alpha]} - 1$, and $r = \frac{\alpha[n+\delta]}{[1-B][1-\alpha]} - \delta$. (See the Appendix for derivations of these Diamond steady state outcomes.) The restriction $\frac{\alpha}{[1-B][1-\alpha]} < 1$ is necessary to obtain the case of interest to Diamond and Tirole, where inefficiency occurs when bubbles do not form. (See the Appendix for the derivation of this condition.) Consequently, in this no bubble steady state, we find $i < 1 + n$ and $r < n + \delta$. The difference between the capital rental rate and the interest rate is $r - i = \frac{[1-\delta][[1-B][1-\alpha]-\alpha]}{[1-B][1-\alpha]}$. Thus, while $i = r$ in the special case of complete depreciation where $\delta = 1$, we find $r > i$ when $0 \leq \delta < 1$.

6 Implications for Interest Rate Theory

Having shown we can obtain a general equilibrium for our model where capital accumulates, and having shown that the interest rate is in general distinct from the capital rental rate, we now summarize the implications of our capital accumulation model for interest rate theory.

First, if depreciation is complete (i.e., $\delta = 1$), then our model with capital accumulation reduces to the Tirole (1985) economy where there is not capital accumulation. In this case, the interest rate is equal to the capital rental rate (i.e., $i_t = r_t$) for all time periods t.

If depreciation is not complete (i.e., $0 \leq \delta < 1$), then the equilibrium path for the interest rate in our model with capital accumulation, like that for the Tirole (1985) model, depends upon whether the bubble converges to its golden rule level or to zero.

If the economy converges to the golden rule, then the equilibrium interest rate and capital rental rate each converge to the growth rate of labor (i.e., $i = r = n$). Thus, as the Tirole (1985) model, the interest rate and capital rental rate for the economy are ultimately determined entirely by the economy's labor growth rate.

However, when depreciation is not complete and the economy is not in the golden rule steady state, the interest rate is never equal to the capital rental rate. When the initial capital level is below the golden rule capital level, we find that the interest rate is always less than the capital rental rate, regardless of whether the economy converges to the golden rule steady state or the diamond steady state. When the initial capital level is above the golden rule level, we find that the interest rate is always greater than the capital rental rate only if the economy converges to the golden rule steady state. In the other cases, where the economy converges to the diamond steady state, the interest rate may be above capital rental rate for some time but then a point is reached where the capital rental rate is above the interest rate thereafter.

In the Diamond steady state, where there is no bubble, whether the interest rate is positive and the extent to which the rental rate on capital exceeds the interest rate depends upon the condition that determines the extent to which a bubble can form. A positive interest rate level requires $\frac{\alpha}{[1-B][1-\alpha]} > \frac{1}{1+n}$. For a bubble to form and to have the case of interest to Diamond (1965) where dynamic inefficiency can occur, the left side of this inequality must be less than one. Thus, the interest rate level will be positive only if the labor growth rate n is high enough. A negative interest rate level will occur if the labor growth rate is close enough to zero. Because the difference between the capital rental rate and the interest rate is $r - i = \frac{[1-\delta][[1-B][1-\alpha]-\alpha]}{[1-B][1-\alpha]}$, we find that the difference between the two in the Diamond steady state is larger when the depreciate rate is further from being complete and when the economy is more susceptible to inefficiency of interest to Diamond.

Except for the depreciation rate, any factor that that increases the marginal product of capital will increase the Diamond steady state interest rate. In particular, as Fisher (1930) contended, an increase in impatience (i.e., an increase in B) will increase the interest rate. Biology still plays a role as emphasized by Samuelson (1958) and Gale (1973) because an increase in the labor growth rate n increases the interest rate. Finally, an increase in the elasticity of output with respect to capital α relative to the increase in the elasticity of output with respect to labor $(1 - \alpha)$ will also increase the interest rate.

7 Conclusion

The overlapping generations models of Diamond (1965) and Tirole (1985) are seminal because they extended the pure consumption model of Samuelson (1958) to include production and production wit bubbles, respectively. However, Tirole followed Diamond, assuming capital does not accumulate but rather is entirely consumed each time period. This paper modifies the Tirole (1985) model so capital accumulates.

When capital accumulates, we show Walras' Law does not hold in general. That is, capital market clearing does not necessarily imply product market clearing. One contribution of this paper is to show that for the original Diamond-Tirole framework, capital market clearing implies product market clearing when the non-depreciated capital is consumed by the old as part of the promised return on saving, but not for two other possible contract arrangements. The more important contribution is to show that, when capital accumulates, capital market clearing implies product market clearing in only two special cases: (1) The case where capital entirely depreciates (i.e., $\delta = 1$), and (2) the case where the economy is in the golden rule steady state (i.e., $r = n$). When capital does not entirely depreciate and the economy is not in the golden rule, we show the interest rate from the economy must deviate from the capital rental rate in order for the product market to clear when the capital market clears.

By allowing the interest rate level to deviate from the capital rental rate, we have provided an updated, general equilibrium OG model with production x==and capital accumulation, where we can delineate the factors that determine the interest rate. While the rate of capital depreciation does affect the capital rental rate, we find it does not affect the steady state interest rate level. If a bubble forms that allows the golden rule steady state to be achieved, then the interest rate level is ultimately determined by the growth rate of labor alone. Alternatively, if the economy converges to the no bubble steady state, the interest rate depends upon a variety of factors, with the interest rate being higher when labor grows faster, when people are more impatient, and when the elasticity of output with respect to capital is higher.

The impact of capital accumulation we have identified and discussed here is important because the overlapping generations model with production is still being usefully applied.[2] It would seem assuming that capital accumulates is more natural than assuming that non-depreciated capital is consumed. Assuming capital is consumed does simplify the model slightly, so scholars may want to continue doing that. However, because we have show that shifting to the assumption that capital accumulates has a significant implication for interest rate determination, it would seem researchers using an OG model with production might fruitfully use a model with capital accumulation.

[2] See Martin and Ventura (2012), Ikeda and Phan (2019), Barbie and Hillebrand (2018), Takao (2019), Brunnermeier and Brunnermeier (2001) for recent significant examples.

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A Mathematical Appendix

A.1 *Finding the Golden Rule Steady State Outcomes*

When we impose stationarity, conditions (4) and (9) yield the consumption possibilities constraint $c^y + \frac{c^o}{[1+n]} = f(k) - [n+\delta]k$ for the representative consumer. In the golden rule steady state, c^y, c^o and k maximize $U(c^y, c^o)$ while satisfying the constraint $c^y + \frac{c^o}{[1+n]} = f(k) - [n+\delta]k$. Optimization with respect to capital implies $f'(k) = n + \delta$. Knowing $r = f'(k) - \delta$ from (5), this last condition implies

$$r = n \tag{A.1}$$

which is one of the key golden rule results. Using the Diamond production function $f(k) = Ak^{\alpha}$, the condition $f'(k) = n + \delta$ implies the golden rule level of capital

$$k = \left[\frac{\alpha A}{n+\delta}\right]^{\frac{1}{1-\alpha}},$$

a second golden rule result.

Using the condition (5) and the Diamond production function $f(k) = Ak^{\alpha}$ to replace $f'(k_t)$ in condition (19) and imposing stationarity, we obtain $1+i = \left[\frac{1+n}{n+\delta}\right] \alpha A k^{\alpha-1}$. Using the golden rule capital stock above to eliminate k, we obtain $1+i = \left[\frac{1+n}{n+\delta}\right]\left[\alpha A \left[\frac{\alpha A}{n+\delta}\right]^{\frac{1}{1-\alpha}}\right]^{\alpha-1}$, which implies

$$i = n, \tag{A.2}$$

a third golden rule result.

Using conditions (1)-(3) together with the Diamond utility function $U(c_t^y, c_{t+1}^o) = Bln(c_t^y) + [1-B]ln(c_{t+1}^o)$, we find that the optimal savings level for the generation t consumer is $[1-B]w_t$. Using conditions (4)-(6) and the Diamond production function $f(k) = Ak^{\alpha}$ to replace w_t, we obtain $s_t = [1-B][1-\alpha]Ak_t^{\alpha}$.

Using this result for the saving level, we can rewrite the capital market clearing condition (8) as $[1-B][1-\alpha]Ak_t^{\alpha} = [1+n]k_{t+1} - [1-\delta]k_t + b_t$. Imposing stationarity, we obtain $[1-B][1-\alpha]Ak^{\alpha} = [1+n]k - [1-\delta]k + b$, which implies $[1-B][1-\alpha]Ak^{\alpha} = [n+\delta]k + b$. Using the golden rule capital stock above to eliminate k, we obtain

$[1-B][1-\alpha]A\left[\frac{\alpha A}{n+\delta}\right]^{\frac{\alpha}{1-\alpha}} = [n+\delta]\left[\frac{\alpha A}{n+\delta}\right]^{\frac{1}{1-\alpha}} + b$, which implies

$$b = [1-B][1-\alpha]A\left[\frac{\alpha A}{n+\delta}\right]^{\frac{\alpha}{1-\alpha}} - [n+\delta]\left[\frac{\alpha A}{n+\delta}\right]^{\frac{1}{1-\alpha}}, \tag{A.3}$$

a fourth golden rule result.

A.2 *Finding the Diamond Steady State Outcomes*

If, as in Diamond (1965) the consumer utility function is $U(c_t^y, c_{t+1}^o) = Bln(c_t^y) + (1-B)ln(c_{t+1}^o)$ and firm production function is $f(k_t) = Ak_t^\alpha$, then assuming $b_t = 0$, the capital market clearing condition (8) becomes $[1-B][1-\alpha]Ak_t^\alpha = [1+n]k_{t+1} - [1-\delta]k_t$. In the steady state, $k_{t+1} = k_t = k$, so this capital market clearing condition becomes $[1-B][1-\alpha]Ak^\alpha + [1-\delta]k = [1+n]k$. Solving, we find the Diamond steady state capital stock level

$$k = \left[\frac{[1-B][1-\alpha]A}{n+\delta}\right]^{\frac{1}{1-\alpha}} \tag{A.4}$$

Knowing $r = f'(k) - \delta$ from condition (5), we obtain $r = \alpha A k^{\alpha-1} - \delta$ when we use the Diamond production function. Using the capital level $(A.4)$, we find the Diamond steady state rental rate on capital is

$$r = \frac{\alpha[n+\delta]}{[1-B][1-\alpha]} - \delta. \tag{A.5}$$

Using the condition (5) and the Diamond production function $f(k) = Ak^\alpha$ to replace $f'(k_t)$ in condition (19) and imposing stationarity, we obtain $1+i = \frac{\alpha A k^{\alpha-1} k}{k - [\frac{1-\delta}{[1+n]}]k}$, which implies $1+i = \frac{[1+n]\alpha A k^{\alpha-1}}{n+\delta}$. Using the Diamond steady state capital stock $(A.4)$ to eliminate k, we obtain $1+i = \frac{[1+n]\alpha}{[1-B][1-\alpha]}$, or

$$i = \frac{[1+n]\alpha}{[1-B][1-\alpha]} - 1 \tag{A.6}$$

A.3 *Proof that* $\frac{\alpha}{[1-B][1-\alpha]} < 1$ *must hold to obtain the case of interest to Diamond (1965)*

The Diamond case holds when the Diamond steady state capital stock level is greater than the golden rule level. In the previous to sections of this appendix, we have shown that the Diamond steady state capital level is $k = \left[\frac{[1-B][1-\alpha]A}{n+\delta}\right]^{\frac{1}{1-\alpha}}$, while the golden rule steady state capital level is $k = \left[\frac{\alpha A}{n+\delta}\right]^{\frac{1}{1-\alpha}}$. Therefore, our condition of interest holds when $\left[\frac{\alpha A}{n+\delta}\right]^{\frac{1}{1-\alpha}} < \left[\frac{[1-B][1-\alpha]A}{n+\delta}\right]^{\frac{1}{1-\alpha}}$, or when $\frac{\alpha}{[1-B][1-\alpha]} < 1$.